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The energy and action of small waves riding on large waves

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We derive the dynamics of small waves riding on larger waves using a canonical, Hamiltonian formulation. The small waves are treated linearly and their energy is derived to all orders in the scale separation between the waves. Our results are similar to those of Longuet-Higgins (1987), but we have extended his calculations to include gravity–capillary waves and to allow for a more general, two-dimensional, large-wave field. Our result for the small-wave Hamiltonian is expressed in both Eulerian (horizontal) coordinate system and a non-inertial system determined by the large wave's surface. On further assuming scale separation between the small and large waves the averaged Lagrangian equations and the action density are derived. Action conservation is explicitly demonstrated.

1. Introduction

Smaller waves on the sea surface are strongly influenced by the larger waves. The spectrum of short waves is expected to vary considerably with the phase of these larger waves. Also the short waves react relatively quickly (compared with the long waves) to the wind, and the wind itself is expected to be affected by the waves. Nonlinear phenomena, such as growth of the wave spectrum and breaking of smaller waves, are certainly influenced by the combined effect of short-wave modulation by the long waves and wind–wave interaction. Among the processes involved in the modulation is the nonlinear interaction between waves of different sizes. It is with this nonlinear interaction that we are concerned in this paper. Other possibly important aspects, such as the generation and decay of short waves, are not studied, so that our considerations are only one aspect of the general problem.

We treat both gravity and capillary waves, as both are believed to be important in the process of wind forcing and wave breaking. We restrict ourselves to a linear treatment of the smaller waves and only comment, where appropriate, on the modifications necessary for a nonlinear treatment. The larger waves are assumed to be fully nonlinear but non-breaking and their complete interaction with the small waves is included. We assume that the large-wave properties in the absence of small waves are given by solutions of the equations of motion. Since the back reaction of the small waves on the large waves is second order in the small-wave amplitude, inclusion of this back reaction in the functions used to describe the large waves will not affect the linear treatment of the small waves, but could be useful in

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a nonlinear extension. On the other hand, calculations of the back reaction *can* be treated with our Hamiltonian, using methods of Whitham (1974*a*) as described briefly in §6.

Throughout the paper the term large waves is meant to refer to waves of large amplitude, while the term small waves refers to a perturbation on those large waves. In most of the paper, no wavelength scale separation is required. When there is a scale separation, large-amplitude long waves are only long in comparison to the small-amplitude short waves. The large waves may, in fact, be capillary waves themselves.

We restrict ourselves to purely irrotational flows, though the extension to wave–current interactions involves similar issues and will be discussed in a future paper.

In the absence of dissipation and forcing, the short waves can be described by wave advection and straining in addition to their intrinsic dynamics, and by conservation of wave action. This problem has been discussed by Garrett & Smith (1976), Phillips (1981) and Longuet-Higgins (1987). Longuet-Higgins dealt with the special case of small waves on a single Stokes wave. One purpose of this paper is to extend the formulation to cover not just a single Stokes wave, but a complete two-dimensional wave field. When we do so, we find exactly the form proposed by Longuet-Higgins for the case of gravity waves on a single Stokes wave.

Garrett & Smith and Longuet-Higgins relied on a change of reference frame from the fixed frame to one accelerating with the fluid surface. In this reference frame the effective gravity is $\boldsymbol{g}' = \boldsymbol{g} - \boldsymbol{a}$, where $\boldsymbol{g}$ is gravity (directed downward), and $\boldsymbol{a}$ is the acceleration of the water at the surface. $\boldsymbol{g}'$ is directed normal to the surface, so that the effective horizontal in the new reference frame lies along the surface. The surface and the acceleration refer to the long wave, as the short wave is only treated linearly. This change of reference frame suffices at the lowest order of the separation of scales between the small wave and the large wave. Although the surface frame is more elegant, in a later section we provide formulas useful for calculations in terms of the horizontal Cartesian coordinates. Since action conservation works better than any order of the scale separation (see, e.g. Landau & Lifshitz 1976), one may want to go beyond lowest order in the expression for the action: there is the worry that there may be additional higher-order terms. These additional terms could involve either the surface curvatures or the rate of tilt of the surface, since neither of these effects occur in a constantly accelerated Cartesian coordinate system. We find an expression for the gravity-wave energy that has no such extra terms. In the capillary–gravity case, there is an extra contribution to effective gravity of $-\sigma(\kappa_1^2+\kappa_2^2)$, where σ is surface tension and κ_1 and κ_2 are the principal curvatures of the long-wave surface.

In the next section we review wave action and the Lagrangian and Hamiltonian descriptions of surface waves. The third section contains the details of the description of the Hamiltonian for small disturbances of a surface-wave field where the position labels are the x, y coordinates on the horizontal plane. The fourth section describes the Hamiltonian for these small waves in a coordinate system moving with the surface of the large waves. We also derive the same equations directly from the Lagrangian. Sections 3 and 4 do not need an assumption about scale separation. The results in §3 are probably of more practical use, while those of §4 are more elegant and may provide a better understanding. In §5 we assume there exists a scale separation; the eikonal (averaged Lagrangian) equation and the appropriate action density are obtained, and action conservation is demonstrated. Section 6 comments briefly on the back reaction of the small waves on the large waves.

2. Action and energy

The principle of action conservation has been known for 120 years (Boltzmann 1866; Clausius 1871). It was expressed in a modern form by Ehrenfest in a number of papers beginning about 75 years ago, Ehrenfest (1959). This principle applies to *any* conservative dynamical system undergoing oscillatory motion. By conservative system is meant one that can be described by a Lagrangian (variational principle), or equivalently, by a Hamiltonian. Only if the Hamiltonian (or equivalent Lagrangian) is time independent is energy conserved. The following results are well established (see e.g. Landau & Lifshitz 1976, and Ehrenfest 1959):

(i) For any periodic conservative system the action is defined to be an area in (p, q) phase space enclosed by the trajectory, i.e. $\oint p\,\mathrm{d}q$, where q is the coordinate, p its conjugate momentum, and the integral is over an entire period. This action is adiabatically invariant, which means that if the parameters describing the oscillator are varied sufficiently slowly the adiabatic invariant changes by an arbitrarily small amount, i.e. this change goes to zero faster than any power of the rate of the parameter variations. There are two caveats in this result: (*a*) the parameter variation must be smooth and (*b*) there is a removable (to all orders) oscillatory piece in the adiabatic invariant that is first order in the parameter change.

(ii) If the Hamiltonian has a kinetic energy T, quadratic in p (but arbitrary in q), and a potential energy V, independent of p, an equivalent expression for the action is $2\langle T\rangle/\omega$, where a time average over a period is taken, and a factor 2π has been left out relative to $\oint p\,\mathrm{d}q$. (Usage in the literature is not consistent on the 2π factor.)

(iii) If a system is linear, there exists a Hamiltonian homogeneous and quadratic in p and q. If the energy E is defined to be the value of this quadratic Hamiltonian, the action can be written as E/ω. (If both (2) and (3) are true, then $\langle V\rangle = \langle T\rangle$.)

(iv) The action is invariant under any canonical transformation. In particular, it is the same in any coordinate system (inertial or non-inertial).

(v) Conservation of action is related to invariance in change of the phase of an oscillation (the relation between an invariance and a conservation law is known as Noether's Theorem).

Whitham introduced these ideas into the study of waves in fluid systems from a somewhat different point of view (see Whitham 1974*a* for a complete presentation). He discovered the equation $\delta\bar{L} = A\,\delta\omega$ connecting the action A, variation in the Lagrangian averaged over a period, and variation in the frequency, ω. He wrote this as $A = \partial\bar{L}/\partial\omega$ which is equal to $\oint p\,\mathrm{d}q$ (Whitham 1974*b*), being undefined when $\delta\omega = 0$. Most of the general properties of action are discussed by Whitham (1974*a*, *b*).

Bretherton & Garrett (1968) consider the special case of those linear waves for which a Lagrangian frame exists, and evaluated E/ω in that frame. (We use the term Lagrangian frame to mean a locally comoving coordinate system which can be considered Cartesian and inertial on the space and time scales of the small waves.) Unfortunately, this paper has led to some misunderstandings owing to the authors requiring that a Lagrangian frame exist and that calculations be done in that frame. The following statements are meant to clear up those misunderstandings.

Well-defined prescriptions exist for calculating the energy E and frequency ω in any frame. Specifically a variational principle can be found by the method of Seliger & Whitham (1968), and E and ω calculated following Whitham (1974*a*, *b*). The action can be evaluated as E/ω in any frame. Whitham (1974*b*) has shown quite generally that although the energy and frequency are frame dependent, their ratio is not. It is

not necessary to work in a Lagrangian frame, in fact a Lagrangian frame may not exist. Even when one does exist, it may be more convenient to use some other frame. For example, in the case of a single long Stokes wave it is convenient to calculate in a frame moving with the phase velocity of the long wave. In that frame all quantities associated with the long wave are time independent, making each of the energy and frequency of the small waves constant.

It is instructive to follow in detail the simple example of surface waves described in two intertial reference frames. The potential energy is the same in both frames, so the total energy changes by as much as the kinetic energy does. The kinetic energy per unit area is

$$KE = \frac{1}{2}\int_{-\infty}^{\zeta} \rho v^2 \,\mathrm{d}z, \tag{2.1}$$

where ζ is the surface elevation. We write $\boldsymbol{v}$ as an advecting part $\boldsymbol{V}$, assumed slowly varying in space, and a wave part $\boldsymbol{v}_{\mathrm{w}}$. We subtract the energy in the absence of the wave, to obtain

$$\begin{aligned} KE &= \int_{-\infty}^{\zeta} (\tfrac{1}{2}\rho V^2 + \rho \boldsymbol{V}\cdot\boldsymbol{v}_{\mathrm{w}} + \tfrac{1}{2}\rho v_{\mathrm{w}}^2)\,\mathrm{d}z - \int_{-\infty}^{0} \tfrac{1}{2}\rho V^2 \,\mathrm{d}z \\ &\approx \tfrac{1}{2}\rho V^2 \zeta + \boldsymbol{V}\cdot\int_{-\infty}^{\zeta} \rho \boldsymbol{v}_{\mathrm{w}}\,\mathrm{d}z + \tfrac{1}{2}\rho \int_{-\infty}^{\zeta} v_{\mathrm{w}}^2 \,\mathrm{d}z. \end{aligned} \tag{2.2}$$

Averaging over a wavelength

$$\langle KE\rangle = \boldsymbol{V}\cdot\boldsymbol{P} + \tfrac{1}{2}\rho\left\langle \int_{-\infty}^{\zeta} v_{\mathrm{w}}^2 \,\mathrm{d}z\right\rangle, \tag{2.3}$$

where the (Stokes drift) momentum is

$$\boldsymbol{P} = \left\langle \int_{0}^{\zeta} \rho \boldsymbol{v}_{\mathrm{w}}\,\mathrm{d}z\right\rangle. \tag{2.4}$$

The change between one reference frame and another moving at velocity $-\boldsymbol{U}$ relative to the first is

$$\delta \boldsymbol{V} = \boldsymbol{U}. \tag{2.5}$$

Therefore

$$\delta E = \delta KE = \boldsymbol{U}\cdot\boldsymbol{P}, \tag{2.6}$$

or

$$E' = E + \boldsymbol{U}\cdot\boldsymbol{P}. \tag{2.7}$$

This transformation, derived for surface water waves, is much more general (see, e.g. Sudarshan & Makunda 1974, Chapter 19, table 1). The invariance of the action and this transformation of the energy can be related as follows: We start with

$$E = A\omega, \quad \boldsymbol{P} = A\boldsymbol{k}, \tag{2.8}$$

connecting the energy and momentum with the action, frequency, and wavenumber. For the last equation see e.g. Phillips (1977, eq. (3.2.13)). The new frequency in the moving frame differs from the old frequency by a Doppler shift

$$\omega' = \omega + \boldsymbol{U}\cdot\boldsymbol{k}, \tag{2.9}$$

so that new energy is

$$E' = A\omega', \tag{2.10}$$

since A is unchanged. Multiplying (2.9) by A, we obtain (2.7), the transformation law for the energy. Since the action is independent of reference frame, we are at liberty to choose any frame that we find convenient to work in, as long as we evaluate both the energy and the frequency in that frame.

We return to our own discussion of Bretherton & Garrett's paper. If a Lagrangian frame exists, it is certainly permissible to evaluate the action in that frame, and then to transform to another frame, such as an Eulerian frame. In general, however, the existence of a Lagrangian frame is not guaranteed, and Bretherton & Garrett's method is inapplicable. There is, however, the general result for linear systems: the energy E is the homogeneous quadratic Hamiltonian, evaluated in any reference frame, and the action is E/ω, where the frequency ω is evaluated in the same reference frame. To lowest order in the scale separation, a Lagrangian frame exists for short waves interacting with long waves. In higher orders, there is not a unique Lagrangian frame, since the current caused by the big wave is depth dependent. In a closely related problem, waves interacting with a current layer, a Lagrangian frame does not even exist in the lowest order. The problem is that there is no locally comoving coordinate system, so one cannot apply the Bretherton–Garrett prescription. The correct action flow velocity has a $\boldsymbol{k}\cdot\partial \boldsymbol{U}(k)\,\partial \boldsymbol{k}$ term in addition to the terms obtained if a Lagrangian frame really existed, where $U(k)$ is an effective, $\boldsymbol{k}$ dependent, velocity. In other words, the current induced shift in the frequency is not a Doppler shift linear in k.

The main calculation that we must perform is to find the energy, i.e. the Hamiltonian, of a small-amplitude wave riding on the larger wave. The small-amplitude assumption is implemented by discarding terms in this Hamiltonian that are cubic or higher in the canonical variables, leaving only the quadratic terms; as discussed above, there are no zero- or first-order terms. The quadratic Hamiltonian is obtained by performing a canonical transformation on the fully nonlinear Hamiltonian for the combination of the big and small waves, followed by a Taylor-series expansion (basic manipulations on Hamiltonians, including canonical transformations, are discussed e.g. in Goldstein 1950). Consider one such canonical transformation. This transformation describes a change of variables which separates each variable of the full problem into the sum of two parts. The first part represents the large waves and is a solution of the equation of motion. The second is the new canonical variable and represents the small waves. No wavelength scale separation is implied at this stage. The remaining ambiguity in the specification of the transformation is resolved by requiring that the new Hamiltonian has no zero-order terms. Since the large wave is a solution of the equations of motion, there will be no first-order terms in the new Hamiltonian. Linear terms in the expansion of the original Hamiltonian are cancelled by terms in the change of the Hamiltonian due to the time derivative of the generating function of the canonical transformation. The time derivative of the generating function is linear, so the quadratic part of the new Hamiltonian is identical with the quadratic part of the expansion of the original energy. Thus in this case the desired Hamiltonian is simply the quadratic term of the expansion of the *original* energy in the small-wave variables. In later sections, we shall combine this transformation with further transformations, whose time dependence does change the value of the quadratic Hamiltonian. They do not, however, change the action. The purpose of these further transformations is to simplify the algebra and the interpretation.

Our starting point is the Hamiltonian for fully nonlinear surface waves. This Hamiltonian, or the related variational principle, has been independently rediscovered by a number of workers: for a review, see Miles (1981). Normally, a Legendre transformation carries one from a variational principle to a Hamiltonian. In the case of fluid problems, however, the constructed variational principle is in canonical form, and it is trivial to read off the Hamiltonian. A Lagrangian in

canonical form (Courant & Hilbert 1937) depends not only on q, but also on p in the following way:

$$L(p_j, q_j; \dot{p}_j, \dot{q}_j) = \sum_j p_j \dot{q}_j - H(p_j, q_j). \tag{2.11}$$

Thus, from a canonical-form Lagrangian it is trivial to read off the p, q pairs of canonical variables and the Hamiltonian $H(p, q)$.

In order to express the surface-wave Hamiltonian in a convenient form, as well as for algebraic convenience, we introduce some notation. We wish to describe waves in terms of variables defined on the water's surface. The element of surface area is

$$\mathrm{d}S = \gamma \,\mathrm{d}x\,\mathrm{d}y, \tag{2.12}$$

with

$$\gamma = (1 + \partial_x \zeta \cdot \partial_x \zeta)^{\frac{1}{2}}, \tag{2.13}$$

where ζ is again the surface elevation. We write $\boldsymbol{x}$ for (x, y) and ∂_x *for* (∂_x, ∂_y). Three-dimensional partial derivatives of, for example, the velocity potential occur in the Hamiltonian, and these need to be expressed in terms of the velocity potential (or other functions) at the surface. Let $f(\boldsymbol{x}, t)$ be any function at the surface. An interior function $g(\boldsymbol{x}, z, t)$ is defined by solving Laplace's equation

$$\left.\begin{aligned} \nabla^2 g &= 0, \\ g(\boldsymbol{x}, \zeta, t) &= f(\boldsymbol{x}, t), \end{aligned}\right\} \tag{2.14}$$

and

$$\hat{\boldsymbol{n}} \cdot \nabla g = 0 \tag{2.15}$$

on the bottom and sides (if any). Thus if f is the velocity potential at the surface, g is the velocity potential in the interior. The operators $\mathbf{D}_x = (\mathrm{D}_x, \mathrm{D}_y)$ and D_z are defined by

$$\mathrm{D}_j f = \partial_j g|_{z=\zeta}. \tag{2.16}$$

Thus, D_j is a linear (but non-local) operator from functions of (x, y) to functions of (x, y). We shall also use the similarly defined normal derivative D_n. When we work on the unperturbed large-wave surface instead of the complete surface, we shall use the notation $\mathbf{D}_{x0}$, D_{z0}, D_{n0} and γ_0. $\mathbf{D}_x$ differs from ∂_x, since

$$\begin{aligned} \partial_x f(\boldsymbol{x}, t) &= \partial_x g[\boldsymbol{x}, \zeta(\boldsymbol{x}, t)\, t] \\ &= \mathbf{D}_x f + (\partial_x \zeta)\, \mathrm{D}_z f. \end{aligned} \tag{2.17}$$

Relations obeyed by the D are

$$\mathbf{D}_x + (\partial_x \zeta)\, \mathrm{D}_z = \partial_x, \tag{2.18}$$

$$\mathrm{D}_z - (\partial_x \zeta) \cdot \mathbf{D}_x = \gamma \mathrm{D}_n, \tag{2.19}$$

and

$$\mathbf{D}_x \cdot \mathbf{D}_x + \mathrm{D}_z^2 = 0. \tag{2.20}$$

The Hamiltonian for surface waves is equal in value to the energy. The canonical coordinates are the surface elevation $\zeta(x, y, t)$ and the velocity potential $\phi(\boldsymbol{x}, z, t)$ evaluated at the surface

$$\phi_{\mathrm{s}}(\boldsymbol{x}, t) = \phi[\boldsymbol{x}, \zeta(\boldsymbol{x}, t), t]. \tag{2.21}$$

The summation in (2.11) is taken to be an integral over $\boldsymbol{x} = (x, y)$. In terms of these variables, the Hamiltonian (West 1981) is

$$H(\zeta, \phi_{\mathrm{s}}) = \tfrac{1}{2}\phi_{\mathrm{s}}(\mathrm{D}_z - \partial_x \zeta \cdot \mathbf{D}_x)\, \phi_{\mathrm{s}} + \tfrac{1}{2} g \zeta^2 + \sigma\gamma. \tag{2.22}$$

We have chosen units in which the density $\rho = 1$. The surface tension is σ. The operators D are functionals of ζ, thereby introducing nonlinearities into H. The

complexity of surface-wave calculations is entirely in dealing with the D operators. By the relations (2.18) and (2.19) only one D operator, for example D_n, is independent, with the others easily obtained by solving (2.18) and (2.19). One method of working with D_n is to expand it in a power series in ζ. For problems involving small waves on big waves, the series cannot be truncated after a few terms, since the ratio of successive terms is of order the product of large-wave amplitude and the small-wave wavenumber; this product is often large.

We write ζ, ϕ_s as

$$\left.\begin{aligned}\zeta &= \zeta_0+\zeta_1,\\ \phi_s &= \phi_{s0}+\phi_{s1},\end{aligned}\right\} \tag{2.23}$$

where the subcript 0 refers to the large wave and 1 to the small wave. We could expand the Hamiltonian in a Taylor series in the 1 variables, retaining the quadratic terms, obtaining the Hamiltonian density H_2. In the next section, we actually obtain an H_2 including a further transformation. In §4, we transform to a coordinate system defined with reference to the unperturbed surface, $z = \zeta_0$. In this system, the meaning of the canonically conjugate variables is different, as the summation of (2.11) is now an integral over the surface area. In this coordinate system the Hamiltonian has the most direct physical interpretation, as given by Longuet-Higgins. In either coordinate system, the new height function must be single valued, which precludes rigorously discussing breaking waves in the x, y coordinates or waves on very curved surfaces in the surface coordinate system. We also carry out the calculation in a very different way, starting with the variational principle and choosing the final variables immediately, thus simplifying the evaluation of the Taylor series.

3. Small-wave dynamics: horizontal coordinates

In this section, the details of the calculation of the Hamiltonian as outlined in §2 are carried out. The starting point of our calculation is the Lagrangian

$$L = \int \mathrm{d}x\{\phi_s \dot{\zeta} - H(\phi_s, \zeta, t)\}, \tag{3.1}$$

where the Hamiltonian density H is given by (2.22). We assume that we have obtained an exact solution with canonical variables ζ_0 and ϕ_{s0} which represents the nonlinear large wave. (As will be shown in §6 this condition can be relaxed to allow small corrections to the equations for ζ_0, ϕ_{s0} quadratic in the small-wave amplitude.) We consider perturbations of this solution by writing

$$\left.\begin{aligned}\zeta(x,y,t) &= \zeta_0+\zeta_1\\ \text{and}\quad \phi_s(x,y,t) &= \phi_{s0}+\phi_{s1},\end{aligned}\right\} \tag{3.2}$$

as in (2.23).

The first step is to find an expression for $D_j\phi_s$ as a perturbation expansion. We shall do this by a Taylor series about the unperturbed surface. Let $\psi(x,y,t)$ be the value of the full velocity potential on the unperturbed surface $z = \zeta_0(x,y,t)$ and use the operators D_{j0}, which are the operators D_j referred to this same unperturbed surface. Then we can write to second order

$$\phi_s = \psi + \zeta_1 D_{z0}\psi + \tfrac{1}{2}\zeta_1^2 D_{z0}^2\psi \tag{3.3}$$

and

$$D_j\phi_s = [1 + \zeta_1 D_{z0} + \tfrac{1}{2}\zeta_1^2 D_{z0}^2] D_{j0}\psi. \tag{3.4}$$

(To be mathematically precise, we must let (3.3) define ψ, since there may be singularities of ϕ between true and unperturbed surfaces when $\zeta < \zeta_0$.) inverting (3.3) we get

$$\begin{aligned}\psi &= [1+\zeta_1 \mathrm{D}_{z0}+\tfrac{1}{2}\zeta_1^2 \mathrm{D}_{z0}^2]^{-1}\phi_{\mathrm{s}} \\ &= \phi_{\mathrm{s}0}+(\phi_{\mathrm{s}1}-\zeta_1 \mathrm{D}_{z0}\phi_{\mathrm{s}0})-\tfrac{1}{2}\zeta_1^2 \mathrm{D}_{z0}^2\phi_{\mathrm{s}0}-\zeta_1 \mathrm{D}_{z0}(\phi_{\mathrm{s}1}-\zeta_1 \mathrm{D}_{z0}\phi_{\mathrm{s}0}).\end{aligned} \tag{3.5}$$

In this expression, notice that $\phi_{\mathrm{s}1}$ always occurs in the combination

$$\psi_{\mathrm{s}1} = \phi_{\mathrm{s}1}-\zeta_1 \mathrm{D}_{z0}\phi_{\mathrm{s}0}, \tag{3.6}$$

which is the part of the total change in ϕ_{s} that arises from changes in the underlying velocity potential and not the part that is the result of evaluating the unperturbed velocity potential at the perturbed surface. It will be useful to recognize this combination when it arises, as in (3.5), where it allows us the slightly more compact form

$$\psi = \phi_{\mathrm{s}0}+(1-\zeta_1 \mathrm{D}_{z0})\psi_{\mathrm{s}1}-\tfrac{1}{2}\zeta_1^2 \mathrm{D}_{z0}^2\phi_{\mathrm{s}0}. \tag{3.7}$$

Now using (3.2) and (3.4) we obtain the expression

$$\phi_{\mathrm{s}} \mathrm{D}_j\phi_{\mathrm{s}} = (\phi_{\mathrm{s}0}+\phi_{\mathrm{s}1}) \mathrm{D}_{j0}\psi+\zeta_1(\phi_{\mathrm{s}0}+\phi_{\mathrm{s}1}) \mathrm{D}_{j0} \mathrm{D}_{z0}\psi+\tfrac{1}{2}\zeta_1^2\phi_{\mathrm{s}0} \mathrm{D}_{j0} \mathrm{D}_{z0}^2\phi_{\mathrm{s}0}, \tag{3.8}$$

where we have used the fact that D_{z0} and D_{j0} commute since the underlying partial derivatives commute with each other and with the Laplacian operator.

The kinetic-energy part of the Hamiltonian (2.22) is

$$T = \tfrac{1}{2}\phi_{\mathrm{s}}\gamma \mathrm{D}_n\phi_{\mathrm{s}} = \tfrac{1}{2}\phi_{\mathrm{s}}[\mathrm{D}_z-(\partial_x\zeta_0)\cdot\mathbf{D}_x-(\partial_x\zeta_1)\cdot\mathbf{D}_x]\phi_{\mathrm{s}}. \tag{3.9}$$

Looking at the third term, we obtain

$$\begin{aligned}-\tfrac{1}{2}(\partial_x\zeta_1)\cdot\phi_{\mathrm{s}}\mathbf{D}_x\phi_{\mathrm{s}} = &-\tfrac{1}{2}(\partial_x\zeta_1)\cdot(\phi_{\mathrm{s}0}+\psi_{\mathrm{s}1})\mathbf{D}_{x0}\psi \\ &-\tfrac{1}{4}(\partial_x\zeta_1^2)\cdot[(\mathrm{D}_{z0}\phi_{\mathrm{s}0})\mathbf{D}_{x0}\phi_{\mathrm{s}0}+\phi_{\mathrm{s}0}\mathbf{D}_{x0} \mathrm{D}_{z0}\phi_{\mathrm{s}0}],\end{aligned} \tag{3.10}$$

where we have chosen to write $\phi_{\mathrm{s}1}$ in terms of $\psi_{\mathrm{s}1}$ using (3.6). At any point in our calculation we could return to $\phi_{\mathrm{s}1}$ and indeed we shall do so when we get our quadratic Hamiltonian. Integration by parts and the use of the chain rule, $\partial_x AB = A\,\partial_x B+(\partial_x A)B$, (which does not hold for the D_j operators) yields for the third term of (3.9)

$$\begin{aligned}&\tfrac{1}{2}\zeta_1[(\partial_x\phi_{\mathrm{s}0}+\partial_x\psi_{\mathrm{s}1})\cdot\mathbf{D}_{x0}\psi+(\phi_{\mathrm{s}0}+\psi_{\mathrm{s}1})\partial_x\cdot\mathbf{D}_{x0}\psi] \\ &\quad+\tfrac{1}{4}\zeta_1^2[(\partial_x \mathrm{D}_{z0}\phi_{\mathrm{s}0})\cdot\mathbf{D}_{x0}\phi_{\mathrm{s}0}+(\mathrm{D}_{z0}\phi_{\mathrm{s}0})\partial_x\cdot\mathbf{D}_{x0}\phi_{\mathrm{s}0}+(\partial_x\phi_{\mathrm{s}0})\cdot\mathbf{D}_{x0} \mathrm{D}_{z0}\phi_{\mathrm{s}0} \\ &\quad+\phi_{\mathrm{s}0}\partial_x\cdot\mathbf{D}_{x0} \mathrm{D}_{z0}\phi_{\mathrm{s}0}].\end{aligned} \tag{3.11}$$

The first two terms of the kinetic energy (3.9) take a similar form:

$$\begin{aligned}\tfrac{1}{2}\phi_s(\mathrm{D}_z-(\partial_x\zeta_0)\cdot\mathbf{D}_x)\phi_{\mathrm{s}} &= \tfrac{1}{2}(\phi_{s0}+\psi_{\mathrm{s}1})\gamma_0 \mathrm{D}_{n0}\psi+\tfrac{1}{2}\zeta_1(\mathrm{D}_{z0}\phi_{\mathrm{s}0})\gamma_0 \mathrm{D}_{n0}\psi \\ &\quad+\tfrac{1}{2}\zeta_1(\phi_{\mathrm{s}0}+\psi_{\mathrm{s}1})\gamma_0 \mathrm{D}_{n0} \mathrm{D}_{z0}\psi+\tfrac{1}{2}\zeta_1^2(\mathrm{D}_{z0}\phi_{\mathrm{s}0})\gamma_0 \mathrm{D}_{n0} \mathrm{D}_{z0}\psi \\ &\quad+\tfrac{1}{4}\zeta_1^2\phi_{\mathrm{s}0}\gamma_0 \mathrm{D}_{n0} \mathrm{D}_{z0}^2\phi_{\mathrm{s}0}.\end{aligned} \tag{3.12}$$

When combining (3.11) and (3.12) it is very convenient to use two identities which follow from (2.18)–(2.20), i.e.

$$\gamma_0 \mathrm{D}_{n0} \mathrm{D}_{z0}A+\partial_x\cdot\mathbf{D}_{x0}A = 0, \tag{3.13}$$

and

$$(\gamma_0 \mathrm{D}_{n0}A)(\mathrm{D}_{z0}B)+(\mathbf{D}_{x0}A)\cdot\partial_x B = (\mathrm{D}_{z0}A) \mathrm{D}_{z0}B+(\mathbf{D}_{x0}A)\cdot\mathbf{D}_{x0}B, \tag{3.14}$$

for any functions A and B. The kinetic energy becomes

$$T = \tfrac{1}{2}(\phi_{s0}+\psi_{s1})\,\gamma_0\,\mathrm{D}_{n0}\,\psi + \tfrac{1}{2}\zeta_1[(\mathrm{D}_{z0}\,\phi_{s0})\,\mathrm{D}_{z0}\,\psi + (\mathbf{D}_{x0}\,\phi_{s0})\cdot\mathbf{D}_{x0}\,\psi + (\partial_x\,\psi_{s1})\cdot\mathbf{D}_{x0}\,\psi]$$
$$+\tfrac{1}{4}\zeta_1^2[(\mathbf{D}_{x0}\,\phi_{s0})\cdot\mathbf{D}_{x0}\,\mathrm{D}_{z0}\,\phi_{s0} + (\mathrm{D}_{z0}\,\phi_{s0})\,\mathrm{D}_{z0}\,\mathrm{D}_{z0}\,\phi_{s0} + (\mathbf{D}_{x0}\,\phi_{s0})\cdot\partial_x\,\mathrm{D}_{z0}\,\phi_{s0}]. \quad (3.15)$$

Finally we make use of the fact that $\gamma_0\,\mathrm{D}_{n0}$ is a Hermitean operator (this follows from the easily verified relation

$$\iint \chi_s^*\,\gamma \mathrm{D}_n\,\psi_s\,\mathrm{d}^2x = \iiint \nabla\chi^*\cdot\nabla\psi\,\mathrm{d}^3x).$$

This allows the second-order pieces of ψ in the first term of (3.15) to be reworked through the use of identity (3.14) so that

$$T = \tfrac{1}{2}(\phi_{s0}+\psi_{s1})\,\gamma_0\,\mathrm{D}_{n0}(\phi_{s0}+\psi_{s1}) + \tfrac{1}{2}\zeta_1[(\mathrm{D}_{z0}\,\phi_{s0})^2 + (\mathbf{D}_{x0}\,\phi_{s0})^2$$
$$+2(\mathbf{D}_{x0}\,\phi_{s0})\cdot\partial_x\,\psi_{s1}] + \tfrac{1}{2}\zeta_1^2(\mathbf{D}_{x0}\,\phi_{s0})\cdot\partial_x\,\mathrm{D}_{z0}\,\phi_{s0}, \quad (3.16)$$

which has the quadratic part

$$T_2 = \tfrac{1}{2}\psi_{s1}\,\gamma_0\,\mathrm{D}_{n0}\,\psi_{s1} + \zeta_1\,\boldsymbol{u}_{0s}\cdot\partial_x\,\psi_{s1} + \tfrac{1}{2}\zeta_1^2\,\boldsymbol{u}_{0s}\cdot\partial_x\,w_{0s}, \quad (3.17)$$

when written in terms of $w_{0s} = \mathrm{D}_{z0}\,\phi_{s0}$ and $\boldsymbol{u}_{0s} = \mathbf{D}_{x0}\,\phi_{s0}$ (respectively, the vertical and horizontal components of the unperturbed fluid velocity evaluated at the unperturbed surface).

The surface-tension term $\sigma\gamma$ of the Hamiltonian (2.22), when expanded in powers of ζ_1, becomes

$$\sigma\gamma = \sigma\left\{\gamma_0 + \frac{(\partial_x\,\zeta_0)\cdot(\partial_x\,\zeta_1)}{\gamma_0} + \frac{\gamma_0^2(\partial_x\,\zeta_1)^2 - (\partial_x\,\zeta_0\cdot\partial_x\,\zeta_1)^2}{2\gamma_0^3}\right\}. \quad (3.18)$$

Combining (3.6), (3.17), (3.18) and the gravitational-potential-energy term we obtain the quadratic Hamiltonian:

$$H_2 = \tfrac{1}{2}(\phi_{s1}-\zeta_1\,w_{0s})\,\gamma_0\,\mathrm{D}_{n0}(\phi_{s1}-\zeta_1\,w_{0s}) + \zeta_1\,\boldsymbol{u}_{0s}\cdot\partial_x(\phi_{s1}-\zeta_1\,w_{0s})$$
$$+\tfrac{1}{2}\zeta_1^2[g+\boldsymbol{u}_{0s}\cdot\partial_x\,w_{0s}] + \frac{\sigma[\gamma_0^2(\partial_x\,\zeta_1)^2 - (\partial_x\,\zeta_0\cdot\partial_x\,\zeta_1)^2]}{2\gamma_0^3}. \quad (3.19)$$

At this point, we have carried out the program outlined in §2. Equation (3.19) suffices as a small-wave Hamiltonian. However, a further canonical transformation simplifies the Hamiltonian. We notice that ϕ_{s1} always appears in the combination that we named ψ_{s1} in (3.6). Therefore, using ψ_{s1} is better than using ϕ_{s1}. In particular, the combination $\mathrm{D}_{n0}(\zeta_1\,w_{0s})$, which has no natural physical interpretation, no longer occurs in H. The desired transformation is generated by

$$G = \zeta_1\,\psi_{s1} + \tfrac{1}{2}\zeta_1^2\,w_{0s}, \quad (3.20)$$

so that $\zeta_1 = \partial G/\partial\psi_{s1}$ and $\phi_{s1} = \partial G/\partial\zeta_1$. Since w_{0s} in G is time dependent, the value of the Hamiltonian changes by $\partial G/\partial t$, and thus the term

$$\frac{\partial G}{\partial t} = \tfrac{1}{2}\zeta_1^2\frac{\partial w_{0s}}{\partial t} = \tfrac{1}{2}\zeta_1^2[(a_z)_{0s} - \boldsymbol{u}_{0s}\cdot\partial_x\,w_{0s}] \quad (3.21)$$

(where $(a_z)_{0s}$ is the acceleration of the unperturbed fluid at its surface) must be added to H_2 in (3.19). The new Hamiltonian is

$$H_2'(\psi_{s1},\zeta_1,t) = \tfrac{1}{2}\psi_{s1}\,\gamma_0\,\mathrm{D}_{n0}\,\psi_{s1} + \tfrac{1}{2}\zeta_1^2\,g_{\mathrm{app}} + \frac{\sigma[\gamma_0^2(\partial_x\,\zeta_1)^2 - (\partial_x\,\zeta_0\cdot\partial_x\,\zeta_1)^2]}{2\gamma_0^3} + \boldsymbol{u}_{0s}\cdot\boldsymbol{P}_2, \quad (3.22)$$

where $$g_{\rm app} = g + (a_z)_{0\rm s} = (\boldsymbol{a}-\boldsymbol{g})\cdot\hat{\boldsymbol{z}}, \tag{3.23}$$

and the momentum density is $$\boldsymbol{P}_2 = \zeta_1 \partial_x \psi_{\rm s1}. \tag{3.24}$$

Our final expression for the quadratic Hamiltonian is thus (3.22) supplemented by definitions (2.13), (2.19), (3.23) and (3.24) (the first two with ζ_0 replacing ζ). The horizontal integral of the first term is equal to the volume integral below the unperturbed surface of $\frac{1}{2}v_1^2$. It is the intrinsic kinetic energy of the small waves. The next term is a modified potential energy, with $g_{\rm app}$ as given by (3.23) replacing g. As will be shown in §4, the somewhat complicated surface-tension term is actually just $\frac{1}{2}(\partial_{\rm s}\zeta_1)^2\sigma/\gamma_0$, where $\partial_{\rm s}$ means the derivative along the unperturbed surface. The last term is the advective term, as in (2.3). It depends on $\boldsymbol{u}_{0\rm s}$, the horizontal fluid velocity at the surface; the vertical fluid velocity does not directly enter.

The equations of motion following from the Hamiltonian (3.22) are

$$\partial_t \zeta_1 = \gamma_0 \mathrm{D}_{n0}\psi_{\rm s1} - \partial_x\cdot(\boldsymbol{u}_{0\rm s}\zeta_1), \tag{3.25}$$

and $$\partial_t \psi_{\rm s1} = -g_{\rm app}\zeta_1 - \boldsymbol{u}_{0\rm s}\cdot\partial_x\psi_{\rm s1} - \sigma\partial_x\cdot\left[\frac{(\partial_x\zeta_0\partial_x\zeta_0\cdot\partial_x\zeta_1 - \gamma_0^2\partial_x\zeta_1)}{\gamma_0^3}\right]. \tag{3.26}$$

4. Small-wave dynamics: surface coordinates

In this section we derive expressions for the dynamics of the small waves in terms of a coordinate system $\boldsymbol{s}$ defined on the unperturbed surface at each instant of time, together with a distance variable n normal to the surface. At one instant of time, the choice of the coordinate system $\boldsymbol{s}$ is arbitrary, but at later times it is fixed by a requirement we impose that a point of fixed $\boldsymbol{s}$ should always move normal to the instantaneous surface. The advantage of this formulation is the ease of interpretation. There are two equivalent ways to do this. In §4.1, the expression for the second-order Hamiltonian H_2 obtained in §3 is modified to accommodate the new variables. In §4.2 an alternative procedure makes the change of variables directly in the original three-dimensional Lagrangian. We conclude this section with the equations of motion for the small waves in terms of surface coordinates.

4.1. *Transformation from horizontal coordinates*

In the expression given in §3 for the second-order Hamiltonian H_2, the functions $\psi_{\rm s1}$ and ζ_1 are thought of as depending on horizontal positions and time. We now consider them as depending on time and on a set of coordinates $\boldsymbol{s}$, which describes the position on the unperturbed surface. The area element of the unperturbed surfaces,

$$\mathrm{d}S = \gamma_0\,\mathrm{d}x\,\mathrm{d}y, \tag{4.1}$$

is assumed expressed in terms of $\boldsymbol{s}$. In §3, the surface functions were considered independent of the vertical z, and for such functions $\boldsymbol{\nabla} = \partial_x$. We now consider them independent of the normal, in which case $\boldsymbol{\nabla} = \partial_{\rm s}$. For functions on the surface, these are related by

$$\partial_{\rm s} = \partial_x - \hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\partial_x), \tag{4.2}$$

where the normal unit vector is

$$\hat{\boldsymbol{n}} = \gamma_0^{-1}(\hat{\boldsymbol{z}} - \partial_x\zeta_0). \tag{4.3}$$

The operator $\partial_{\rm s}$ is a gradient on three-dimensional physical space, and should not be confused with $(\partial_{\rm s1}, \partial_{\rm s2})$, the gradient in two-dimensional $\boldsymbol{s}$-space. We also define the

perturbed surface in terms of normal distances $h(\boldsymbol{s},t)$ from the unperturbed surface, in which case

$$\zeta_1 = \gamma_0 h(\boldsymbol{s},t) \tag{4.4}$$

to lowest order. The change is necessary because h and ϕ_{s1} are the canonical variables in these coordinates.

The gradient of the pressure, $\boldsymbol{g}-\boldsymbol{a}$, has a dynamically determined normal part and a tangential part determined by surface tension. Thus, we have

$$(\boldsymbol{a}-\boldsymbol{g}) = \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}\cdot(\boldsymbol{a}-\boldsymbol{g}) - \sigma\,\partial_s(\kappa_1+\kappa_2), \tag{4.5}$$

where κ_1, κ_2 are the principal surface curvatures. The second-order part of the surface tension energy in (3.22) becomes much simpler in these variables since

$$(\partial_s\zeta_1)^2 = (\partial_x\zeta_1)^2 - \frac{(\partial_x\zeta_0\cdot\partial_x\zeta_1)^2}{\gamma_0^2}, \tag{4.6}$$

and

$$(\partial_s\zeta_1)^2 = (\gamma_0\partial_s h)^2 + h^2(\gamma_0^2\partial_s\gamma_0^{-1})^2 - (\gamma_0^3\partial_s\gamma_0^{-1})\cdot(\partial_s h^2). \tag{4.7}$$

An integration by parts on the $\boldsymbol{s}$ variables combines the terms of (4.7) into

$$\int \gamma_0^{-2}(\partial_s\zeta_1)^2\,\mathrm{d}S = \int\{(\partial_s h)^2 + h^2\gamma_0\partial_s^2\gamma_0^{-1}\}\,\mathrm{d}S. \tag{4.8}$$

This is expressed in geometric terms by means of

$$\gamma_0\partial_s^2\gamma_0^{-1} = -\kappa_1^2 - \kappa_2^2 - \gamma_0(\partial_x\zeta_0)\cdot\partial_s(\kappa_1+\kappa_2). \tag{4.9}$$

The last terms in (4.9) are seen to cancel the contribution from the last term of (4.5) by using (4.3) to express $\partial_x\zeta_0$ in terms of $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{z}}$.

Finally we note that the time-dependent change from integration variables $\boldsymbol{x}$ to surface variables $\boldsymbol{s}$ affects what we mean by the partial time derivative of any function F,

$$\left.\frac{\partial F}{\partial t}\right|_{x\,\mathrm{fixed}} = \left.\frac{\partial F}{\partial t}\right|_{s\,\mathrm{fixed}} + \left.\frac{\partial \boldsymbol{s}}{\partial t}\right|_{x\,\mathrm{fixed}}\cdot\frac{\partial F}{\partial \boldsymbol{s}}, \tag{4.10}$$

which shifts part of the term $\psi_{s1}\dot{\zeta}_1$ into the Hamiltonian

$$\zeta_1\dot{\psi}_{s1}|_x = \zeta_1\dot{\psi}_{s1}|_s + \zeta_1\left.\frac{\partial \boldsymbol{s}}{\partial t}\right|_x\cdot\frac{\partial\psi_{s1}}{\partial \boldsymbol{s}}. \tag{4.11}$$

Putting (4.1)–(4.11) into (3.22) we get

$$L = \int \mathrm{d}S\{-h\dot{\psi}_{s1} - H''(h,\psi_{s1},t)\}, \tag{4.12}$$

with the quadratic part of H'' having the form

$$H_2'' = \tfrac{1}{2}\{\psi_{s1}\,\mathrm{D}_{n0}\,\psi_{s1} + g_{\mathrm{eff}}h^2 + \sigma(\partial_s h)^2\} + \boldsymbol{U}\cdot\boldsymbol{P}_s, \tag{4.13}$$

where

$$g_{\mathrm{eff}} = (\boldsymbol{a}-\boldsymbol{g})\cdot\hat{\boldsymbol{n}} - (\sigma)\,(\kappa_1^2+\kappa_2^2), \quad \boldsymbol{P}_s = h\,\partial_s\psi_{s1}. \tag{4.14}$$

Here $\boldsymbol{U}$ is the unperturbed velocity of the fluid *along* the unperturbed surface (not the same as $\boldsymbol{u}_{0s}$ which is horizontal) and $\boldsymbol{P}_s$ is the wave momentum in the surface coordinate system. Comparison of (4.13) with the original Hamiltonian, (2.22), shows that the only changes are an advection along the instantaneous unperturbed surface and an effective gravity which depends on both the unperturbed acceleration normal to the surface and the principal curvatures of the surface. All other curvature effects are contained in the variables $\boldsymbol{s}$.

The equations of motion derived from (4.13) are presented in §4.3.

4.2. *An alternative approach*

We now derive the above result by an alternative technique, starting from the Lagrangian in the form given by Luke (1967), with surface tension added:

$$-L = \iiint^{\zeta} \left[\frac{\partial \phi}{\partial t} + \tfrac{1}{2}(\nabla\phi)^2 + gz\right] \mathrm{d}\boldsymbol{x}\,\mathrm{d}z + \sigma S, \tag{4.15}$$

where S is the excess surface area (over the horizontal projection). We want to refer the variables of the perturbed motion, to the unperturbed surface Σ, which we assume is represented by some time-dependent parametrization, with parameters $\boldsymbol{s} = (s_1, s_2)$. These parameters together with a coordinate n, measuring the position of a point along a normal $\hat{\boldsymbol{n}}$ to Σ ($n = 0$ on Σ), serve as a coordinate system. It can only be a proper coordinate system in a neighbourhood of Σ. (At a distance given by the smaller of the radii of curvatures of Σ, caustics start developing. We only need the coordinate system a distance of the order of the perturbation amplitude away from Σ, so this is a negligible restriction.)

We write

$$\phi = \phi_0 + \phi_1, \tag{4.16}$$

where ϕ_0 represents the unperturbed flow, and ϕ_1 is the perturbation. The perturbed surface is described relative to Σ by

$$n = h(\boldsymbol{s}, t), \tag{4.17}$$

where $|h|$ measures its distance from Σ along a normal $\hat{\boldsymbol{n}}$ to the latter. Expanding $L = L_0 + L_1 + L_2 + \ldots$ in ϕ_1 and h, L_0 describes the unperturbed flow and L_1 can be shown to vanish as a consequence of the stationarity of L_0. L_2 gives the linearized equations by requiring that L_2 be stationary with respect to variations of h and ϕ_1. It is readily seen that L_2 is to be extracted from the following part of L:

$$\iiint_{V_1} \left[-p_0 + \frac{\partial \phi_1}{\partial t} + \nabla\phi_0 \cdot \nabla\phi_1\right] \mathrm{d}V + \iiint_{V_0} \tfrac{1}{2}(\nabla\phi_1)^2\,\mathrm{d}V + \sigma S, \tag{4.18}$$

where V_1 is the domain enclosed between the perturbed surface and Σ, and V_0 is the domain below Σ, $\mathrm{d}V$ is the volume element, and p_0 is the unperturbed pressure given by

$$-p_0 = \frac{\partial \phi_0}{\partial t} + \tfrac{1}{2}(\nabla\phi_0)^2 + gz. \tag{4.19}$$

The second-order contribution from the first term in the first integral is

$$-\frac{1}{2}\iint_{\Sigma} \left[\left.\frac{\partial p_0}{\partial n}\right|_{\Sigma} - p_{0s}(\kappa_1 + \kappa_2)\right] h^2\,\mathrm{d}S, \tag{4.20}$$

where $\mathrm{d}S$ is an area element of Σ, and $\partial p_0/\partial n|_{\Sigma}$ is the component of the unperturbed pressure gradient normal to Σ. Finally $\kappa_1 + \kappa_2$ is twice the mean curvature of Σ. The second-order contribution from the second and third terms in the first integral of expression (4.18) is

$$\iint_{\Sigma} \left[\frac{\partial \phi_1}{\partial t} + \nabla\phi_0 \cdot \nabla\phi_1\right]_{\Sigma} h\,\mathrm{d}S. \tag{4.21}$$

We want to express the integral in terms of $\psi_{s1} = \phi_1|_{\Sigma}$. Since the time derivative occurs, we have to fix our ideas of the time-dependent parametrization of Σ:

$$\boldsymbol{r}(\boldsymbol{s}, t) = \{\boldsymbol{x}(\boldsymbol{s}, t), \zeta_0[\boldsymbol{x}(\boldsymbol{s}, t), t]\}. \tag{4.22}$$

We denote by V and U the components of the unperturbed surface velocity, $\nabla\phi_0|_\Sigma$, perpendicular and parallel to Σ, respectively. It is convenient to restrict the possible representations to those satisfying

$$\frac{\partial \boldsymbol{r}}{\partial t} = V, \tag{4.23}$$

i.e. a point $\boldsymbol{s}$ = constant, sitting on Σ, is moving perpendicular to Σ.

The velocity relative to a point on Σ of constant $\boldsymbol{s}$ is $\boldsymbol{U}$, so we write

$$\left[\frac{\partial \phi_1}{\partial t} + \nabla\phi_0 \cdot \nabla\phi_1\right]_\Sigma = \frac{\partial \psi_{s1}}{\partial t} + \boldsymbol{U}\cdot\partial_s \psi_{s1}, \tag{4.24}$$

where ψ_{s1} depends on $\boldsymbol{s}$ and t through (4.21) and ∂_s is the gradient operator on the surface Σ (as defined by (4.2)). Here $\partial/\partial t$ means the rate of change at a surface point moving perpendicular to Σ ($\boldsymbol{s}$ = constant). The second-order contribution to σS becomes

$$\sigma \iint_\Sigma [\kappa_G h^2 + \tfrac{1}{2}(\partial_s h)^2]\,\mathrm{d}S, \tag{4.25}$$

where $\kappa_G = \kappa_1\kappa_2$ is the Gaussian curvature of the unperturbed surface Σ.

Assembling terms we have

$$-L_2 = \iint_\Sigma \left[h\frac{\mathrm{d}\psi_{s1}}{\mathrm{d}t} + \left(\sigma\kappa_G + \tfrac{1}{2}p_{0s}(\kappa_1+\kappa_2) - \frac{1}{2}\frac{\partial p_0}{\partial n}\bigg|_\Sigma\right)h^2 + \tfrac{1}{2}\sigma(\partial_s h)^2\right]\mathrm{d}S,$$

$$+ \iiint_{V_0} \tfrac{1}{2}(\nabla\phi_1)^2\,\mathrm{d}V, \tag{4.26}$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{U}\cdot\partial_s$. Since atmospheric pressure can be arbitrarily set to zero, the surface pressure p_{0s} is given by $-\sigma(\kappa_1+\kappa_2)$. Using this we obtain

$$-L_2 = \iint_\Sigma \left[h\frac{\mathrm{d}\psi_{s1}}{\mathrm{d}t} + \tfrac{1}{2}g_{\mathrm{eff}}h^2 + \tfrac{1}{2}\sigma(\partial_s h)^2\right]\mathrm{d}S + \iiint_{V_0} \tfrac{1}{2}(\nabla\phi_1)^2\,\mathrm{d}V. \tag{4.27}$$

Here g_{eff} is defined in (4.14) above, and we have used the fact that

$$-\frac{\partial p_0}{\partial n}\bigg|_\Sigma = (\boldsymbol{a}-\boldsymbol{g})\cdot\hat{\boldsymbol{n}}. \tag{4.28}$$

From (4.26) the linearized equations can be obtained by variation with respect to h and γ_1. In particular we obtain the equation of continuity

$$\nabla^2\phi_1 = 0. \tag{4.29}$$

Taking (4.29) as a constraint, the last integral of (4.26) is readily transformed to a surface integral, and the Lagrangian reads

$$L_2 = \iint_\Sigma \left\{-h\frac{\partial\psi_{s1}}{\partial t} - \tfrac{1}{2}[\psi_{s1}\,\mathrm{D}_{n0}\psi_{s1} + g_{\mathrm{eff}}h^2 + \sigma(\partial_s h)^2] - h\boldsymbol{U}\cdot\partial_s\psi_{s1}\right\}\mathrm{d}S$$

$$= \iint_\Sigma \left[-h\frac{\partial\psi_{s1}}{\partial t} - H\right]\mathrm{d}S. \tag{4.30}$$

Here we have used the linear operators D_{n0} with the property $\mathrm{D}_{n0}\psi_{s1} = \partial\phi_1/\partial n|_\Sigma$. We have further transformed the first term in (4.26) to an equivalent form (the

difference being terms of the form $\partial(\ )/\partial t$ and $\partial_s\cdot(\ )$). The Hamiltonian density H is then given by

$$H = \tfrac{1}{2}\{\psi_{s1}\,\mathrm{D}_{n0}\,\psi_{s1} + g_{\mathrm{eff}}\,h^2 + \sigma(\partial_s h)^2\} + h\boldsymbol{U}\cdot\partial_s\psi_{s1}, \tag{4.31}$$

which is the same as (4.13), where h and ψ_{s1} are the generalized coordinate and its conjugate variable respectively.

4.3. *Interpretation and equations of motion*

The Hamiltonians (4.13) and (4.31) are very similar to the Hamiltonian for linear waves on a flat surface. The first term is the kinetic energy, which for linear waves on a flat surface is $\frac{1}{2}\phi_s\,\mathrm{D}_z\,\phi_s$. The actual term reflects the tilt of the surface by big waves. The surface integral of this term is equal to the volume integral of $\frac{1}{2}v_1^2$. (We assume no flow through the bottom or side boundaries.) This term in the integrand differs from the corresponding term in (3.22) by a factor γ_0, the change in the integration measure. The second term is a modified gravitational potential energy. In the pure-gravity-wave case, the modification simply reflects the accleration of the surface fluid. With surface tension, $\boldsymbol{a}-\boldsymbol{g}$ is not normal to the surface, and the effective gravity (4.14) contains a surface-tension contribution. In the pure-gravity-wave case, this term differs from the corresponding term in (3.22) by the same factor γ_0 that the first term does. The surface-tension term is much simpler in surface coordinates than in horizontal coordinates, and is just the linear wave surface-tension energy tilted by the big wave. The last term is the advective term of the expected form $\boldsymbol{U}_s\cdot\boldsymbol{P}_s$, where $\boldsymbol{U}_s$ is the large-wave fluid velocity parallel to the surface and $\boldsymbol{P}_s$ is the small-wave momentum parallel to the surface. It is to be contrasted with the corresponding term in (3.22), $\boldsymbol{u}_{0s}\cdot\boldsymbol{P}_2$, involving horizontal velocity and momentum. These corresponding terms are not in a ratio of γ_0 unless the big-wave flow is steady. This is because the value of the energy changes between the description in the two different coordinate systems for a time-dependent transformation.

The surface-tension part of the perturbed Lagrangian is $\frac{1}{2}\sigma[(\partial_s h)^2-(\kappa_1^2+\kappa_2^2)h^2]$. This is identical with the expression one would get for the perturbed potential energy of a surface film (e.g. soap film) perturbed from a form where the local principal curvatures are κ_1 and κ_2. Consider, the for example, a cylindrical film of radius R. An axial perturbation with a wavelength longer than the circumference $2\pi R$ gives a negative perturbed potential energy, and thus leads to instability. (It is well known that cylindrical surface films with lengths longer than the circumference are unstable.)

The linear equations of motion obtained by variation of (4.31) with respect to h and ψ_{s1}, taking account of the time dependence of $\mathrm{d}S$, are

$$\frac{\partial h}{\partial t} + (\kappa_1+\kappa_2)\,h\,\hat{\boldsymbol{n}}\cdot\boldsymbol{V} = \frac{\delta H}{\delta\psi_{s1}} = \mathrm{D}_{n0}\,\psi_{s1} - \partial_s\cdot(h\boldsymbol{U}), \tag{4.32}$$

and

$$\frac{\partial\psi_{s1}}{\partial t} = -\frac{\delta H}{\delta h} = -g_{\mathrm{eff}}\,h + \sigma(\partial_s^2 h) - c\boldsymbol{U}\cdot\partial_s\psi_{s1}. \tag{4.33}$$

5. The action

In the two previous sections there was no assumption of scale separation in wavelength between the perturbed and the unperturbed motions. We now turn our attention to the case where the perturbation represents a short wave, i.e. its

wavelength is assumed short compared to a characteristic scale of the unperturbed motion. It is in this case where the action of the short wave is expected to be an adiabatic invariant. Calculations assuming scale separation are often much better than might be expected. For example Furry (1946) found, when the 'large' and 'small' scales were essentially equal, that the quantity that he was calculating was only wrong by 13%. Action conservation is also expected to work very well even when scales are not too disparate. For action conservation to work to some order, it is necessary that the energy be calculated properly to that order. Low-order oscillating parts of the simple action we calculate can be removed either analytically (Kruskal 1962) or by averaging in space and or in time.

In the first part of this section we derive the equations of motion for the action where the coordinate system is tied to the longer waves. In the last part we derive the equations appropriate to a fixed, flat space coordinate system.

Because the expression for the Hamiltonians in §3 or 4 are similar to the linear wave Hamiltonian, the action is the same as the linear action with possibly four changes. The first change is that g is replaced by either g_{eff} for the surface coordinate Hamiltonian or g_{app} for horizontal coordinates. Other changes for only the surface coordinate expressions are that the amplitude a is measured normal to the surface (not vertically), and the wavenumber k is measured along the surface (not horizontally). The last change is that D_{n0} is not equivalent to k, except in lowest order. For completeness, we show these results.

We assume the perturbation to look locally like a plane wave and to be represented by (in the surface coordinate system, with trivial changes in the horizontal coordinate system)

$$\left.\begin{aligned} \psi_{\text{s1}} &= C\sin\chi, \quad h = a\cos\chi, \\ \boldsymbol{k} &= \partial_{\text{s}}\chi, \quad \omega = -\frac{\partial\chi}{\partial t} \end{aligned}\right\} \tag{5.1}$$

where

are the local wave vector and frequency, respectively. C, a, ω and $\boldsymbol{k}$ are assumed to be slowly varying functions of t and position on Σ. Using scale separation the relations between ω, k, C and a can be found by either of two related procedures: (i) inserting (5.1) into (3.22) or (4.31) and using standard WKB techniques; or (ii) using Whitham's method (see Whitham 1974*a*, *b*), i.e. by inserting (5.1) into (4.30) or the Lagrangian obtained from (3.22), finding the phase-averaged Lagrangian density $\bar{L}$ and varying with respect to a and C. This method carries over to nonlinear short waves. In either case we find the relations

$$\left.\begin{aligned} &(\omega-\boldsymbol{k}\cdot\boldsymbol{U})\,C-(g_{\text{eff}}+\sigma k^2)\,a=0, \\ \text{and}\quad &(\omega-\boldsymbol{k}\cdot\boldsymbol{U})\,a-d_{n0}\,C=0, \end{aligned}\right\} \tag{5.2}$$

where d_{n0} is the appropriate eigenvalue of the operator D_{n0}. (The horizontal-coordinate-system relations differ from (5.2) only by geometrical factors.)

5.1. *Surface coordinates*

For surface coordinates, d_{n0} has the form

$$d_{n0} = k+\tfrac{1}{2}\kappa_{\perp}+O\left(\frac{1}{k}\right), \tag{5.3}$$

where the curvature perpendicular to $\boldsymbol{k}$ is

$$\kappa_{\perp} = \kappa_1\sin^2\theta_1+\kappa_2\cos^2\theta_1, \tag{5.4}$$

and θ_1 is the angle between $\boldsymbol{k}$ and the direction of the first principal curvature. In what follows, we approximate d_{n0} by its first term k; more accurate expressions are easily constructed. The dispersion relation following from (5.2) is then

$$(\omega - \boldsymbol{k}\cdot\boldsymbol{U})^2 = g_{\text{eff}}\, k + \sigma k^3, \tag{5.5}$$

which, in the gravity-wave case, was given by Garrett & Smith (1976).

Using (5.2), the phase-averaged Lagrangian and Hamiltonian densities read

$$\bar{L} = \frac{a^2}{4k}[(\omega - \boldsymbol{k}\cdot\boldsymbol{U})^2 - g_{\text{eff}}\, k - \sigma k^3], \tag{5.6}$$

and

$$\bar{H} = \frac{a^2}{2k}\omega(\omega - \boldsymbol{k}\cdot\boldsymbol{U}) = \omega\frac{\partial \bar{L}}{\partial \omega}. \tag{5.7}$$

The action density is now given by

$$A = \frac{E}{\omega} = \frac{\partial \bar{L}}{\partial \omega} = \frac{a^2}{2k}(\omega - \boldsymbol{k}\cdot\boldsymbol{U}) = \tfrac{1}{2}a^2\left(\frac{g_{\text{eff}}}{k} + \sigma k\right)^{\frac{1}{2}}, \tag{5.8}$$

where $E = \bar{H}$. Variation of the averaged Lagrangian $\bar{L}$ with respect to the phase χ then gives (see Whitham 1974*a*)

$$\frac{\partial}{\partial t}\left(\frac{\partial \bar{L}}{\partial \omega}\right) - \partial_{\mathrm{s}}\cdot\left(\frac{\partial \bar{L}}{\partial \boldsymbol{k}}\right) = 0, \tag{5.9}$$

which by (5.8) is readily transformed to

$$\frac{\partial A}{\partial t} + \partial_{\mathrm{s}}\cdot\left(A\frac{\partial \omega}{\partial \boldsymbol{k}}\right) = 0, \tag{5.10}$$

giving the proper conservation equation for the action. Note that, with our definition of ω in (5.5), the group velocity, $\boldsymbol{V}_{\mathrm{g}} = \partial\omega/\partial\boldsymbol{k}$, is tangent to the surface.

The eikonal equations describing the motion (in three-dimensional space) of a wave packet of small, short waves are readily obtained. Denoting $\boldsymbol{V}_{\mathrm{n}}$ as the large-wave velocity normal to the surface these equations are

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{V}_{\mathrm{g}}(x,t) + \boldsymbol{V}_{\mathrm{n}}(x,t), \tag{5.11}$$

$$\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} + \partial_{\mathrm{s}}|_k\,\omega = -\hat{\boldsymbol{n}}\left(\frac{\mathrm{d}\hat{\boldsymbol{n}}}{\mathrm{d}t}\cdot\boldsymbol{k}\right) - |\boldsymbol{V}_{\mathrm{n}}|\sum_j \boldsymbol{\nabla}(\hat{n}_j)\, k_j, \tag{5.12}$$

where it is understood that $\boldsymbol{k} \equiv \boldsymbol{k}(x(t),t)$ and $\partial_{\mathrm{s}}|_k\,\omega = [\boldsymbol{\nabla} - \hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla})]|_k\,\omega$ means that the spatial derivative is taken, keeping $\boldsymbol{k}(x(t),t)$ fixed. The term $\boldsymbol{V}_{\mathrm{n}}$ in (5.11) ensures that the wave packet stays on the surface of the large wave. For a flat surface the right-hand side of (5.12) vanishes yielding the usual eikonal equation (following from the integrability condition for (5.1)). Extra terms arise in the integrability condition on a curved surface because $\mathrm{d}/\mathrm{d}t$ and ∂_{s} are not gradients (i.e. they do not commute). The two extra terms present for a curved surface have a simple interpretation. The first ensures that the wave vector $\boldsymbol{k}$ remains tangent to the surface (i.e. $\hat{\boldsymbol{n}}\cdot\boldsymbol{k} = 0$). The second term is due to wave stretching. Our definition of the time derivative requires that 'positions' of the surface move only normal to the surface, which implies that the distance between points on a curved surface changes, i.e. the surface 'stretches'. The second term ensures that, as the surface 'stretches', the number of wavelengths

(of the small waves) between two points on the surface would not change in the absence of motion relative to the surface coordinates. This term is also necessary to guarantee the covariance of (5.11) and (5.12), as otherwise (5.12) would change under a transformation from one inertial reference frame to another.

5.2. *Horizontal coordinates*

For practical calculations the surface coordinate system is awkward to use. We therefore sketch the derivation of the ray equations for the flat space coordinate system.

The dispersion relation requires evaluating the operator $\gamma_0 \mathrm{D}_{n0}$ on a sinusoidal function $\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$. We define $d_{n0}(\boldsymbol{x},\boldsymbol{k})$ by

$$\gamma_0 d_{n0} = \mathrm{Re}\,[\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\gamma_0 \mathrm{D}_{n0}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}]. \tag{5.13}$$

On a flat horizontal surface it would have the value k. We have evaluated the first two terms of γd_{n0} in powers of $1/k$. This is accomplished by using an exact solution of Laplace's equation

$$f = \int \mathrm{d}^2\boldsymbol{q}_\mathrm{s}\,\mathrm{e}^{\mathrm{i}\boldsymbol{q}_\mathrm{s}\cdot\boldsymbol{t}}\mathrm{e}^{q_\mathrm{s}n}G(\boldsymbol{q}_\mathrm{s}), \tag{5.14}$$

where $\ln G(\boldsymbol{q}_\mathrm{s})$ is an arbitrary quadratic function of $\boldsymbol{q}_\mathrm{s}$. The vector $\boldsymbol{t}$ is the projection of the position $\boldsymbol{r}$ onto the plane tangent to the wave and n is the distance to the plane. The parameters of G are chosen to make

$$\ln f|_{z=\zeta_0} = \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} + O(x^3) \quad \text{as } x\to 0.$$

Then

$$\gamma_0 d_{n0} \approx \mathrm{Re}\,[\gamma_0 \partial_n f|_{z=\zeta_0}]. \tag{5.15}$$

When this is evaluated, we find

$$\gamma_0 d_{n0} = k\mu + \left(\frac{\nabla^2\zeta_0 - \hat{\boldsymbol{k}}\cdot\nabla\nabla\zeta_0\cdot\hat{\boldsymbol{k}}}{2\mu^2}\right) + O\left(\frac{1}{k}\right), \tag{5.16}$$

with μ defined by

$$\mu^2 = \gamma_0^2 - (\hat{\boldsymbol{k}}\cdot\nabla\zeta_0)^2. \tag{5.17}$$

The quantity γ_0/μ is the ratio of the wavenumber to its projection onto the plane tangent to the surface. The correction term to $\gamma_0 d_{n0}$ involves the second derivative of ζ_0, related to the curvature, in the direction perpendicular to $\boldsymbol{k}$.

The small-wavelike approximation for (3.25) and (3.26) follows from these equations by the replacements

$$\partial_t \to -\mathrm{i}\omega, \tag{5.18}$$

$$\partial_x \to \mathrm{i}k, \tag{5.19}$$

$$\gamma_0 \mathrm{D}_{n0} \to \gamma_0 d_{n0}. \tag{5.20}$$

They can be solved for ω, giving

$$\omega = [\gamma_0 d_{n0}(g_\mathrm{app} + \sigma k^2\mu^2\gamma_0^{-3})]^{\frac{1}{2}} + \boldsymbol{u}_{0\mathrm{s}}\cdot\boldsymbol{k}. \tag{5.21}$$

The ray equations are the usual

$$\dot{\boldsymbol{x}} = \frac{\partial\omega}{\partial\boldsymbol{k}}, \tag{5.22}$$

$$\dot{\boldsymbol{k}} = -\frac{\partial\omega}{\partial\boldsymbol{x}}. \tag{5.23}$$

The action has the same form as in (5.8)–(5.10), taking $\gamma_0 d_{n0} = k\mu$,

$$A = \frac{b^2}{2\gamma_0}\left(\frac{g_{\text{app}}}{k\mu/\gamma_0} + \sigma k\mu/\gamma_0\right)^{\frac{1}{2}} \tag{5.24}$$

$$\frac{\partial A}{\partial t} + \partial_x \cdot \left(A \frac{\partial \omega}{\partial \boldsymbol{k}}\right) = 0 \tag{5.25}$$

but the symbols b and k now refer to vertical height and horizontal wavenumber.

6. Radiation reaction on the large waves

In the preceding sections, we have treated the large waves as comprising an exact solution of the equations of motion without the small waves. We mentioned in the introduction that such an assumption is not required for the calculation that we did, but that the back reaction was allowed in the large waves, because it was of higher order in the small-wave amplitude than we included. Clearly, there is a certain amount of choice in the definition of the large waves.

If one chooses the large wave to be an exact solution, the canonical transformation (2.23) eliminates the term in the Hamiltonian linear in the small wave, but makes no other changes. Thus, not only the quadratic, but all higher terms are unchanged by that transformation. What should be the back reaction occurs in the cubic terms, and the small-wave variables evolve to include a contribution that would be better thought of as a radiation-reaction contribution to the large waves.

If, on the other hand, the big waves are chosen to respond dynamically to the radiation pressure from the small waves, there is still a certain amount of ambiguity in the choice of the decomposition into large and small, $\Psi = \Psi_L + \Psi_S$, where Ψ represents the pair (ζ, ϕ_s). One particularly attractive resolution of the ambiguity is to choose

$$L(\Psi_L, \Psi_S) = \tfrac{1}{2}[L_{\text{old}}(\Psi_L + \Psi_S) + L_{\text{old}}(\Psi_L - \Psi_S)]. \tag{6.1}$$

The equations of motion are

$$\tfrac{1}{2}[\delta L_{\text{old}}(\Psi_L + \Psi_S) \pm \delta L_{\text{old}}(\Psi_L - \Psi_S)] = 0. \tag{6.2}$$

The sum of these two equations is the original equation of motion and the difference is the dynamical equation for the decomposition. (Clearly, any choice for the function of $\Psi_L - \Psi_S$ would also yield this property. Our particular choice is motivated by Whitham's averaging technique. If Ψ_S is to represent a single short wave, the average of L over the short-wave wavelength eliminates all terms odd in Ψ_S.) In a Taylor-series expansion of L_{old} in powers of Ψ_S, all odd terms are discarded to obtain the Lagrangian. The cubic term is not present, and the quadratic term, (3.22), thought of as a function of the large-wave variables, provides the correction to the unperturbed Hamiltonian which gives the radiation reaction.

If one makes the short-wavelength approximation described in §5, the back reaction can be conveniently described in terms of the energy and the action of the small waves. The entire Hamiltonian must be expressed in a consistent representation. Since the large waves are best described in the horizontal coordinate system of §3, the small-wave energy and frequency are to be evaluated in this coordinate system. One treats the large-wave variables dynamically, rather than specified in advance as we have. One writes the Hamiltonian of the entire system as

$$H = H_0(\zeta_0, \phi_{s0}) + \sum_{\text{small waves}} A\omega, \tag{6.3}$$

and treats A as a constant. The back reaction is included in the equations of motion

$$\dot{\zeta}_0 = \frac{\partial H_0}{\partial \phi_{s0}} + \Sigma A \frac{\partial \omega}{\partial \phi_{s0}}, \tag{6.4}$$

and

$$-\dot{\phi}_{s0} = \frac{\partial H_0}{\partial \zeta_0} + \Sigma A \frac{\partial \omega}{\partial \zeta_0}. \tag{6.5}$$

The variations of ω with respect to the fields $\phi_{\sigma 0}, \zeta_0$ have delta functions at the positions of the small waves. That (6.3) can be interpreted as a Hamiltonian can be derived by the method of Whitham (1974*a*).

7. Discussion

We have derived the dynamics of small waves riding on larger waves using a canonical formulation. The straining and advection is caused by the large waves, which can be fully nonlinear. The small waves are treated linearly and their dynamics remains such in the presence of larger waves. The modifications due to the large waves are a tilting of the reference surface, a modified gravity, and an advective term. These results are similar to those of Longuet-Higgins. We have extended his calculations to include gravity–capillary waves and to allow for a more general large wave (instead of a single Stokes wave). This allows the application of our results to a realistic ocean surface. We also show that there is no ambiguity in the calculation of wave action. Our final result for the small-wave Hamiltonian can be expressed in two equivalent forms: in terms of horizontal coordinates (3.22) or in terms of surface coordinates (4.31). For the pure-gravity case, the action obtained from this latter expression is identical with that of Garrett & Smith and of Longuet-Higgins when the wave amplitude is interpreted as normal to the surface and the wavenumber is evaluated parallel to the surface.

In the derivation of the appropriate short-wave equations of motion, we assumed that the larger waves were exact solutions to the original equations. However, our procedure is also appropriate for approximate solutions to these equations. That is, we pretend we have exact solutions until we obtain the quadratic Hamiltonian. Otherwise, perturbation terms appear which try to correct the approximate solution and are not related to the short-wave dynamics. Proper treatment of conserved quantities, such as wave action, shows that the large-wave solutions should be treated as exact. However, this is necessary only to first order in the short-wave amplitude and this is sufficient to calculate the reaction of the small waves back on the large waves. The back reaction of the small waves on the large waves is quadratic in the small wave amplitude, and therefore does not modify the small wave equation to the order we are following. Indeed, techniques exploiting scale separation are a very convenient way of calculating the back reaction of small, short waves on large, longer waves.

In summary, we have calculated the energy and action of small waves riding on large waves. This calculation has been done in a horizontal coordinate system in which the different terms in the energy are *not* obtained by transforming free waves to a different coordinate system. Any calculations or simulations of the behaviour of small waves on any but the simplest long wave (e.g. a Stokes wave) are most conveniently done in this coordinate system. In order to gain some intuition into the meaning of the energy terms and to compare to previous work, the Hamiltonian was transformed into the non-inertial frame moving with the surface of the long wave.

Because of mathematical complications, it is difficult to perform calculations using this non-inertial frame. On the other hand, the expressions in this frame are useful in showing that the consequence of the long wave is to give an effective gravity and to advect the small wave. On further assuming a scale separation, the averaged Lagrangian equation and action density were derived and action conservation was explicitly demonstrated.

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